Expansion of a Gas Cloud by the Discrete Velocity Method

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A discrete velocity model is presented to reduce the Boltzmann equation into a set of equations which are used to solve the unsteady expansion of a rarefied ($K_n = 0.1 - \infty$) gas cloud bounded by a vacuum. Good agreement is observed among results derived by the present method and those of the free molecular flow method. The discrete velocity method is to divide the moving directions of the molecules into finite (here, we take two) groups, then a discrete velocity model is formed based on this division. In this way the effect of molecular collisions is taken into account.

Nomenclature

C = mean thermal speed of molecules
f = distribution function
K_n = Knudsen number
N = gas density
n = density in dimensionless form
R = spatial coordinate in dimensionless form
r = spatial coordinate
S = collision cross section
T = time in dimensionless form
t = time
u = mean velocity in dimensionless form
V = mean gas velocity
σ = dimensional index (0, 1, 2)
λ = mean free path

Subscripts

1 = leftward
2 = rightward
0 = initial
ℓ = leading edge
t = trailing edge

Introduction

THE unsteady expansion of a gas cloud into a vacuum is a basically interesting problem in the fields of gasdynamics and astrophysics. Molmud[1] presented the density distribution of free molecular flow when a gas cloud bounded by a vacuum is expanding. Mirels and Mullen[2] obtained results of the expansion of a dense gas cloud. Broadwell[3,4] introduced a discrete velocity method. Later, Gatignal,[5] Cabannes,[6] Harris,[7] and Godunov[8] investigated this method. The purpose of this paper is to present a discrete collision model based on Broadwell's method and use this model to obtain the density and velocity distribution of an unsteady expanding rarefied gas cloud bounded by a vacuum.

Discrete Velocity Method for Gas Expansion

We know that molecules in a gas cloud can move in any direction. If the probability of molecules moving in any direction is the same, the gas cloud is in a thermal equilibrium state. If the probability of molecules moving in one direction increases while the probability of molecules moving in other directions decreases, the gas is in unsteady motion. We divide all the possible directions into finite (for example, m) groups. Let f_i be the probability of molecules moving in direction $i (i = 1,2.,,,.m)$, then

$$f = \Sigma f_i \qquad i = 1,2,\ldots,m \tag{1}$$

Actually, the probability of molecules moving in driection i represents the density of the gas moving in this direction. Therefore, we use density N_i instead of f_i. The Boltzmann equation can be written in the form

$$\frac{\partial N_i}{\partial t} + \frac{\partial (V_i N_i)}{\partial r_i} + \frac{\sigma V_i N_i}{r_i} = \left(\frac{\partial f}{\partial t}\right)_{ci} \tag{2}$$

where $(\partial f/\partial t)_{ci}$ is the discrete collision term, N_i the density of gas molecules moving in direction i, and V_i the velocity of gas molecules moving in direction i. If in some way we derived N_i, the gas density N could be found[3,4] by

$$N = \Sigma N_i \tag{3}$$

and the mean gas velocity could be found by

$$V = \Sigma (V_i N_i)/N \tag{4}$$

This is the basic idea of the discrete method.

For the symmetric expansion problem, we can use the simple two-direction model to derive the collision terms. No matter which direction the molecules move in, we can say the motion is either leftward (direction 1) or rightward (direction 2). N_1 and N_2 are the densities of molecules 1 (leftward) and molecules 2 (rightward), respectively (Fig. 1). We assume that the collisions between molecules are perfectly elastic. All of the molecules move at constant speed C ($|V_i| = C$, C is taken as the mean thermal speed). The frequency of collisions between N_1 and N_2 is $SC_rN_1N_2$ where $C_r = 2C$ is the relative speed. We take N_2 as an example to illustrate the change of N_2 due to collisions. The frequency of collisions can be rewritten as $SC_rN_1[N_1 + (N_2 - N_1)]$. That means the collisions between molecules 1 and 2 are divided into two parts: One of them is $SC_rN_1N_1$, which has no effect on the change of density and, hence, will not be taken into account. The other one is $SC_rN_1(N_2 - N_1)$, which represents effective collisions. Every effective collision produces a new molecule 2, so that the change of N_2 due to effective collisions is $SC_rN_1(N_2 - N_1)$. When molecule 2 is produced, molecule 1 disappears. Consequently, Eq. (2) can be written as a set of equations

$$\frac{\partial N_1}{\partial t} + \frac{\partial (V_1 N_1)}{\partial r_1} + \frac{\sigma V_1 N_1}{r_1} = -SC_rN_1(N_2 - N_1) \tag{5}$$

$$\frac{\partial N_2}{\partial t} + \frac{\partial (V_2 N_2)}{\partial r_2} + \frac{\sigma V_2 N_2}{r_2} = SC_rN_1(N_2 - N_1) \tag{6}$$

These are the governing equations for the expansion by the discrete velocity method. If we take N_1 as an example, noting that the r_i, N_i, and V_i are in the same direction, the resulting equations will be the same as Eqs. (5) and (6).

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Unsteady Expansion of a Gas Cloud into a Vacuum

Suppose there is a gas cloud distributed uniformly within a region bounded by an impermeable shell. At the beginning, the shell is removed instantaneously and the gas is allowed to expand freely into a vacuum. It could be plane, cylindrical, or spherical expansion. Let r_0 be the initial gas boundary position. During expansion, there is a leading edge of expansion moving outward from r_0, and a trailing edge moving inward (Fig. 2). The position of the leading edge is $r_\ell = r_0 + Ct$, and the position of the trailing edge is $r_t = r_0 - Ct$. When the trailing edge arrived at the center of the gas cloud, a reflected trailing edge would be generated and move outward. We assume that there is no outside force acting on the molecules and no outside energy being added to the molecules. The initial gas density is assumed to be N_0 with K_n ranging from 0.1 to ∞ [$K_n = \lambda/r_0$, where $\lambda = (\sqrt{2}SN_0)^{-1}$ is the initial mean free path]. Let the direction of r (Fig. 2) be the direction 2. Then, $V_1 = -C$, $V_2 = C$, $r_2 = r$, $\Delta r_2 = \Delta r$, $r_1 = -r$, $\Delta r_1 = -\Delta(-r) = \Delta r$. Multiply Eqs. (5) and (6) by r_0/CN_0 and let $n_i = N_i/N_0$, $T = tC/r_0$, $R = r/r_0$. The equations for expansion of a gas cloud can be rewritten as follows:

$$\frac{\partial n_1}{\partial T} - \frac{\partial n_1}{\partial R} + \frac{\sigma n_1}{R} = \frac{\sqrt{2}}{K_n} n_1 (n_1 - n_2) \tag{7}$$

$$\frac{\partial n_2}{\partial T} + \frac{\partial n_2}{\partial R} + \frac{\sigma n_2}{R} = \frac{\sqrt{2}}{K_n} n_1 (n_2 - n_1) \tag{8}$$

The initial and boundary conditions are

$$\begin{aligned} &T=0, \quad R \geq R_0, && n_1 = n_2 = 0 \\ &\qquad 0 \leq R < R_0, && n_1 = n_2 = \tfrac{1}{2} \\ &T>0, \quad R=0, && \left.\frac{\partial n}{\partial R}\right|_{R=0} = 0 \\ &\qquad R \geq R_\ell, && n_1 = n_2 = 0 \end{aligned} \tag{9}$$

where R_ℓ is the position of the leading edge in dimensionless form, $R_\ell = 1 + T$. Using Eqs. (7), (8), and initial and boundary conditions Eq. (9), we obtained the numerical solutions for n_1 and n_2 varying with time T and position R.

The density n and mean velocity u are found from Eqs. (3) and (4) to be

$$n = n_1 + n_2 \tag{10}$$

$$u \equiv V/C = (n_2 - n_1)/n \tag{11}$$

The results are presented by the plots on n vs R and the plots of u vs R at different times in Figs. 3 and 4.

Results and Discussion

1) The difference between the free molecular flow method and the present method is that the effect of collisions on the change of density and velocity have been taken into account by the present method. If the $K_n \to \infty$, the collision term is zero. Then, the equations will be written as

$$\frac{\partial n_1}{\partial T} - \frac{\partial n_1}{\partial R} + \frac{\sigma n_1}{R} = 0 \tag{12}$$

$$\frac{\partial n_2}{\partial T} + \frac{\partial n_2}{\partial R} + \frac{\sigma n_2}{R} = 0 \tag{13}$$

Using the same initial and boundary conditons as Eq. (9), the numerical results of Eqs. (12) and (13) have been obtained for free molecular expansion. Comparing with Molmud's results[1] (Fig. 5), we can see good agreement between our results and Molmud's results of the density distribution.

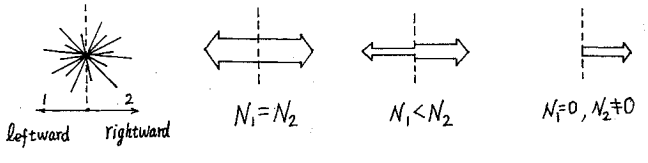


Fig. 1 Two directions model for gas expansion.

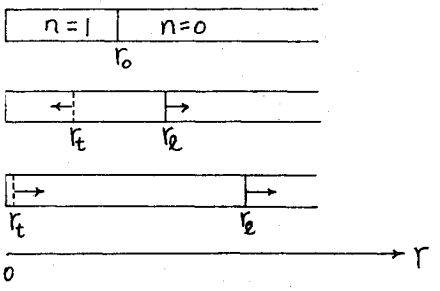
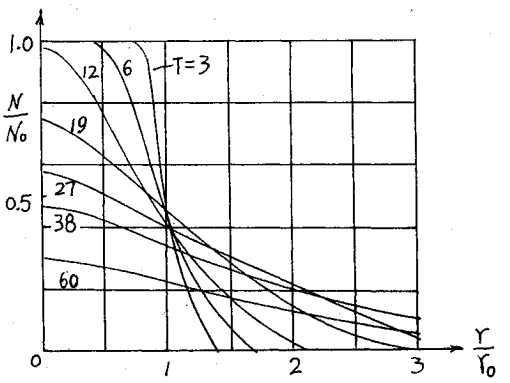
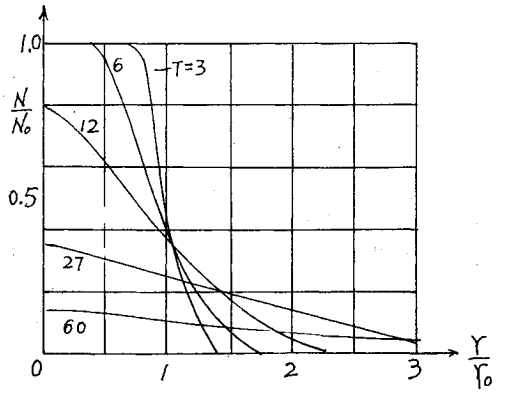


Fig. 2 Unsteady expansion of a gas cloud.





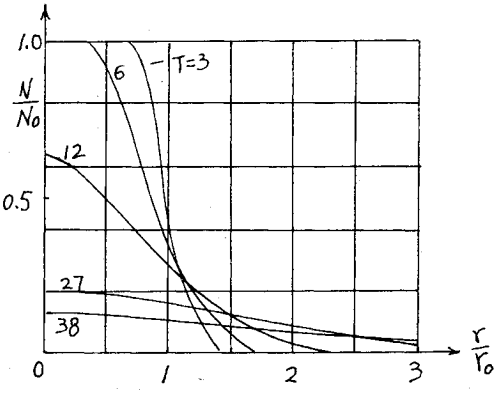


Fig. 3 The density distribution of a gas cloud expanding into vacuum, $K_n = 1.0$, $T = tC/r_0$; a) plane expansion, b) cylindrical expansion, and c) spherical expansion.

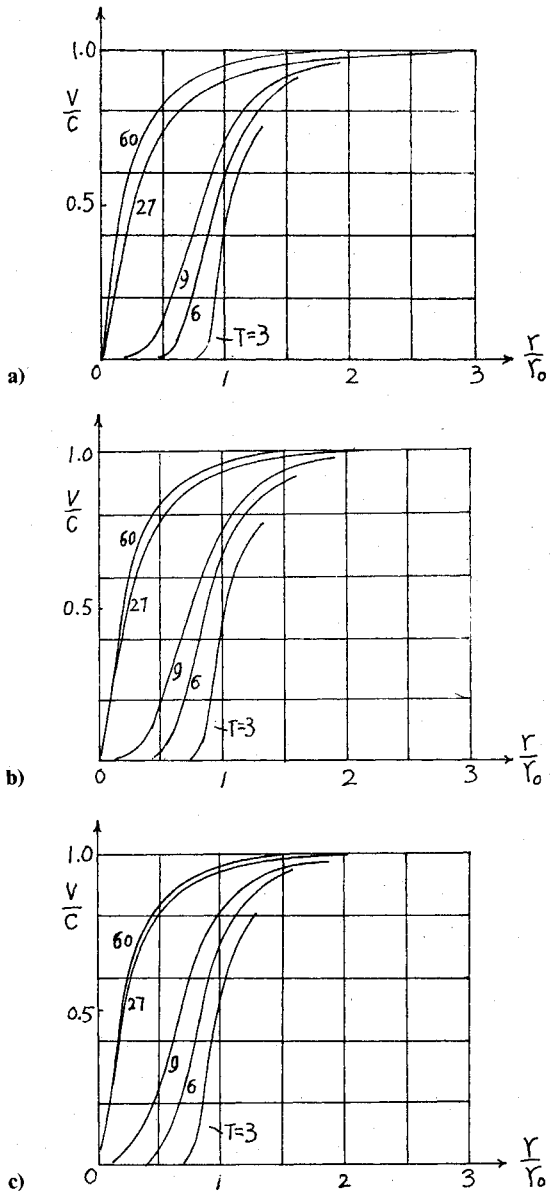


Fig. 4 The velocity distribution of a gas cloud expanding into vacuum, $K_n = 1.0$, $T = tC/r_0$; a) plane expansion, b) cylindrical expansion, and c) spherical expansion.

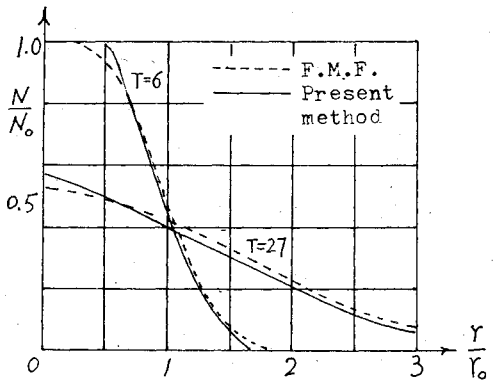


Fig. 5 Comparison between the present method and the free molecular flow method in density distribution, $K_n \to \infty$.

2) We also obtained the distributions of density and velocity for different K_n numbers. The density distributions for $K_n = 0.1$ and $K_n = 100$ are shown in Fig. 6; the density distribution for $K_n = 1.0$ is in between. From the different results due to different K_n numbers, we can see that, in the early stage of expansion, the effect of collisions is to accelerate the expansion; later on, the effect of collisions is to decelerate the expansion. Generally, the effect of variation of K_n on the expansion is not significant.

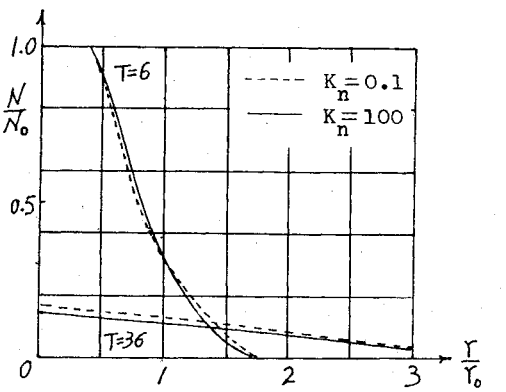


Fig. 6 The effect of K_n number on the density distribution (spherical expansion).

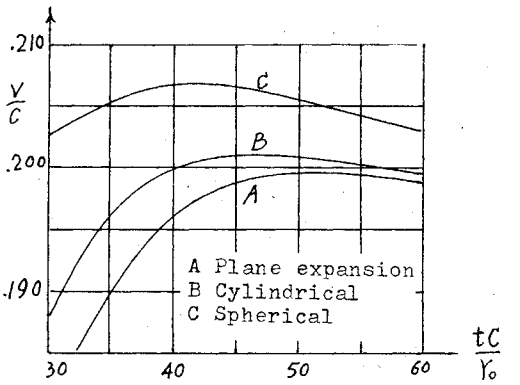


Fig. 7 The enlarged detail of variation of gas velocity at the point near symmetric center, K_n 1.0.

3) The mean velocity of gas at the point of symmetric center is always zero. At a point other than the symmetric center the gas velocity increases with time to a maximum, then decreases slowly. For example, the velocity distribution with time at the point near the symmetric center is plotted in Fig. 7.

Finally, the discrete method is an approximate method to solve the Boltzmann equation. The assumption that all of the molecules move at constant speed needs to be improved. Nevertheless, this method appears to be a promising one.

Acknowledgment

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